Quadratic and Cubic ODE systems: Fitzhugh-Nagumo, van der Pol, Izhikevich

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Problem 1: Non linear oscillators (van der Pol): Derive two forms The Fitzhugh-Nagumo equation system, which was derived to explore the properties of the Hodgkin Huxley system using qualitative dynamical systems theory, is basically a version of the van der Pol oscillator, so it is useful to be familiar with the van der Pol oscillator.

The equation system, in second order is $\ddot{x} \ = \ -\mu(1-x^2)\dot{x} \ -x$ where μ is a parameter. Note that if $\mu = 0$, then this is a simple harmonic osciallator with $\ddot{x} \ = \ -x$ which has periodic solutions like $\exp \pm it$.

The two matlab programs on the homework site vdp_phase_plot_non_cubic.m and vdp_phase_plot_cubic.m will be of help.

1. The first task will be to understand the ode system. Starting with the second order expression for the van der Pol oscillator, rewrite it as two first order systems using the methods we discussed in class. You should end up with

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} \ = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} \ = \ \begin{pmatrix} \dot{x} \\ -\mu(1-x^2)\dot{x} - x) \end{pmatrix} \tag{0.1}$$

To put this in matlab friendly notation, let the vector $\vec{y} = \binom{y1}{y2}$ and show you get the following (matlab like) system.

$$\frac{d}{dt}\begin{pmatrix} y1 \\ y2 \end{pmatrix} \ = \ \begin{pmatrix} y2 \\ -\mu(1-y1^2)y2 - y1) \end{pmatrix} \tag{0.2}$$

Now, run the code vdp_phase_plot_non_cubic.m. Note that this is not the cubic system like Fitzhugh Nagumo, but is the most direct way to write the van der Pol equation, as outlined above – this is the "cookbook" way to reduce a high order ode to a system of first order.

Run the code with the paramter NORMALIZE = 1 and then NORMALIZE = 0 (the parameter NORMALIZE is on line 9. Explain qualitatively the properties of the phase portrait and of the trajectories of the system in terms of the nullclines given by $\dot{y_1} = 0$ and $\dot{y_2} = 0$. Explain what the point of this NORMALIZE parameter is, and why it might be useful.

2. The previous excercise did NOT reference a cubic null-cline, which we need to understand the Fitzhugh-Nagumo system, which is based on the properties of a cubic null-cline. So lets consider the "Lienard Transform" way of writing the van der Pol system So, let's start with the following "matlab like" system:

$$\frac{d}{dt}\begin{pmatrix} y1 \\ y2 \end{pmatrix} = \begin{pmatrix} \dot{y1} \\ \dot{y2} \end{pmatrix} = \begin{pmatrix} y2 - \mu G(y1) \\ -y1 \end{pmatrix} \tag{0.3}$$

where $G(y1) = \frac{y1^3}{3} - y1$. Show that this is the same van der Pol system as the previous method. In other words, show that this system, written in terms of a cubic polynomial, is equivalent to the van der Pol equation $\ddot{x} = -\mu(1-x^2)\dot{x} - x$.

Hint: you can do this easily by taking the first derivative of the equation $\dot{y1} = y2 - \mu G(y1)$ and using $\dot{y2} = -y(1)$ – the trick is to write it all in terms of $y1$. You will then have the van der Pol equation $\ddot{x} = -\mu(1-x^2)\dot{x} - x$ in terms of $y1$ – show this in your homework.

Now, run the code vdp_phase_plot_cubic.m (from the homework site).

3. Run the code with the parameter NORMALIZE = 1 and then NORMALIZE = 0 (the parameter NORMALIZE is on line 9). Explain qualtitatively the properties of the phase portrait and of the trajectories of the system in terms of the nullclines given by $\dot{y1} = 0$ and $\dot{y2} = 0$.

Problem 2: Modify the Fitzhugh-Nagumo model starting from vanderPol model

This problem is to show a bit more of the relation of the van der Pol model and the Fitzhugh-Nagumo model.

1. Run the code (from the homework site) fitz_nag_els_phase_plot.m and compare the output from running the code vdp_phase_plot_cubic.m. Note that the vdp oscillator produces a more regular output – it is what we expect from an oscillator. But the fitz_nag code is less evenly spaced, more like what we expect from generating "spikes". So, make a simple modifiction of the vdp code to make it look a bit more like the fitz_nag code (hint: change the relative scale of y1,y2).

2. Now, use the original fitz_nag code to show the phenomenon of "Anodal break". This refers to the observation that a strong negative voltage step (strongly inhibitory) can (seemingly paradoxically) elicit a single spike. You can do this with the fitz_nag code by changing the initial condition in the code to a much more negative voltage starting point (so, produce a figure of a strongly negative starting voltage resulting in a single spike). Explain "anodal break" in terms of the Hodgkin Huxley system.

Izhikevich Quadratic Integrate and Fire Model With reference to the figure (on the homework site) izhi_phase_plot_fig.jpg, trace out the movement of the system starting from the point [-50,-20] with update rules: $if v > 30 : v\_reset \leftarrow v, u+10 \leftarrow u$. Try v_reset = -50 and d = 2. (Note, you can reproduce this figure by running the code izhi_phase_plot.m (and note the NORMALIZE flag in that code!). The code izhikevich.m on the homework site produces a gui that implements this system – you can play with that a bit if you want. (The 2003 paper by Izhikevich which describes this model is on the web site. It's worth while reading it, since it is very short and at this point, the system he uses should be well understood).